Stationary isothermic surfaces and some characterizations of the hyperplane in the N-dimensional Euclidean space*

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Abstract

We consider an entire graph S of a continuous real function over $\mathbb{R}^{N-1}$ with $N \geq 3$. Let Ω be a domain in $\mathbb{R}^N$ with S as a boundary. Consider in Ω the heat flow with initial temperature 0 and boundary temperature 1. The problem we consider is to characterize S in such a way that there exists a stationary isothermic surface in Ω. We show that S must be a hyperplane under some general conditions on S. This is related to Liouville or Bernstein-type theorems for some elliptic Monge-Ampère-type equation.

Key words. Heat equation, overdetermined problems, stationary isothermic surfaces, hyperplanes, Monge-Ampère-type equation.

AMS subject classifications. Primary 35K05, 35K20, 35J60; Secondary 35J25.

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1 Introduction

Let Ω be a domain in $\mathbb{R}^N$ with $N \geq 3$, and let $u = u(x,t)$ be the unique bounded solution of the following problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}. \tag{1.3}$$

The problem we consider is to characterize the boundary $\partial\Omega$ in such a way that the solution u has a stationary isothermic surface, say Γ. A hypersurface Γ in Ω is said to be a *stationary isothermic surface* of u if at each time t the solution u remains constant on Γ (a constant depending on t). It is easy to see that stationary isothermic surfaces occur when $\partial\Omega$ and Γ are either parallel hyperplanes, concentric spheres, or coaxial spherical cylinders. The level surfaces of u then are the so-called *isoparametric surfaces* whose complete classification in Euclidean space was given by Levi-Civita [LC] and Segre [Seg].

Almost complete characterizations of the sphere have already been obtained by [MS1, MS2] with the help of Aleksandrov's sphere theorem [Alek]. In [MS2], we also derived some characterizations of the hyperplane mainly based on geometrical arguments: under suitable global assumptions on $\partial\Omega$, if Ω contains a stationary isothermic surface, then $\partial\Omega$ must be a hyperplane. In the present paper, we produce new results in this direction mainly based on partial differential equations techniques (in Section 3, we compare them to the ones obtained in [MS2]). Assume that Ω satisfies the uniform exterior sphere condition and Ω is given by

$$\Omega = \{\ x = (x', x_N) \in \mathbb{R}^N \ :\ x_N > \varphi(x')\ \}, \tag{1.4}$$

where $\varphi = \varphi(x')$ ($x' \in \mathbb{R}^{N-1}$) is a continuous function on $\mathbb{R}^{N-1}$. We recall that Ω satisfies the *uniform exterior sphere condition* if there exists a number $r_0 > 0$ such that for every $\xi \in \partial\Omega$ there exists an open ball $B_{r_0}(y)$, centered at $y \in \mathbb{R}^N$ and with radius $r_0 > 0$, satisfying $\overline{B_{r_0}(y)} \cap \overline{\Omega} = \{\xi\}$.

We state our main result.

Theorem 1.1 *Assume that there exists a stationary isothermic surface* $\Gamma \subset \Omega$*. Then, under one of the following conditions* (i), (ii), *and* (iii)*,* $\partial\Omega$ *must be a hyperplane.*

(i) $N = 3$;

(ii) $N \geq 4$ *and* φ *is globally Lipschitz continuous on* $\mathbb{R}^{N-1}$;

(iii) $N \geq 4$ *and there exists a non-empty open subset* A *of* $\partial\Omega$ *such that on* A *either* $H_{\partial\Omega} \geq 0$ *or* $\kappa_j \leq 0$ *for all* $j = 1, \cdots, N-1$.

(Here $H_{\partial\Omega}$ *and* $\kappa_1, \cdots, \kappa_{N-1}$ *denote the mean curvature of* $\partial\Omega$ *and the principal curvatures of* $\partial\Omega$*, respectively, with respect to the upward normal vector to* $\partial\Omega$*.)*

Remark. When $N = 2$, this problem is easy. Since the curvature of the curve $\partial\Omega$ is constant from (2.3) in Lemma 2.1 in Section 2 of this paper, we see that $\partial\Omega$ must be a straight line.

Also, notice that, if φ is either convex or concave, then (iii) is surely satisfied.

2 A proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. Let $d = d(x)$ be the distance function defined by

$$d(x) = \operatorname{dist}(x, \partial\Omega), \quad x \in \Omega. \tag{2.1}$$

We start with a lemma.

Lemma 2.1 *The following assertions hold:*

(1) $\Gamma = \{ (x', \psi(x')) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1} \}$ *for some real analytic function* $\psi = \psi(x')$ $(x' \in \mathbb{R}^{N-1})$*;*

(2) *There exists a number $R > 0$ such that $d(x) = R$ for every $x \in \Gamma$;*

(3) *φ is real analytic, the mapping: $\partial\Omega \ni \xi \mapsto x(\xi) \equiv \xi + R\nu(\xi) \in \Gamma$ ($\nu(\xi)$ denotes the upward unit normal vector to $\partial\Omega$ at $\xi \in \partial\Omega$) is a diffeomorphism, and $\partial\Omega$ and Γ are parallel hypersurfaces at distance R;*

(4) *the following inequality holds: for each $j = 1, \cdots, N-1$*

$$-\frac{1}{r_0} \le \kappa_j(\xi) < \frac{1}{R} \quad \text{for every } \xi \in \partial\Omega, \tag{2.2}$$

where $r_0 > 0$ is the radius of the uniform exterior sphere for Ω;

(5) *there exists a number $c > 0$ satisfying*

$$\prod_{j=1}^{N-1} \left(\frac{1}{R} - \kappa_j(\xi) \right) = c \quad \text{for every } \xi \in \partial\Omega. \tag{2.3}$$

Proof. The strong maximum principle implies that $\frac{\partial u}{\partial x_N} < 0$, and (1) holds. Since Γ is stationary isothermic, (2) follows from a result of Varadhan [Va]:

$$-\frac{1}{\sqrt{s}} \log W(x, s) \to d(x) \ \text{ as } s \to \infty,$$

where

$$W(x, s) = s \int_0^\infty u(x, t) e^{-st} dt \ \text{ for } s > 0. \tag{2.4}$$

The inequality $-\frac{1}{r_0} \le \kappa_j(\xi)$ in (2.2) follows from the uniform exterior sphere condition for Ω. See [MS2, Lemma 2.2] together with [MS1, Lemma 3.1] for the remaining claims. □

With the help of Lemma 2.1, we notice that φ is an entire solution over $\mathbb{R}^{N-1}$ of the elliptic Monge-Ampère-type equation (2.3). Thus, Theorem 1.1 is related to Liouville or Bernstein-type theorems.

Let us proceed to the proof of Theorem 1.1. Set

$$\Gamma^* = \left\{ x \in \Omega : d(x) = \frac{R}{2} \right\}. \tag{2.5}$$

Denote by κ_j^* and $\hat\kappa_j$ $(j = 1, \cdots, N-1)$ the principal curvatures of Γ^* and Γ, respectively, with respect to the upward unit normal vectors. Then, the mean curvatures H_{Γ^*} and H_Γ of Γ^* and Γ are given by

$$H_{\Gamma^*} = \frac{1}{N-1}\sum_{j=1}^{N-1}\kappa_j^* \quad \text{and} \quad H_\Gamma = \frac{1}{N-1}\sum_{j=1}^{N-1}\hat\kappa_j,$$

respectively. These principal curvatures have the following relationship: for each $j = 1, \cdots, N-1$,

$$\kappa_j(\xi) = \frac{\kappa_j^*(\xi^*)}{1+\frac{R}{2}\kappa_j^*(\xi^*)} = \frac{\hat\kappa_j(\hat\xi)}{1+R\hat\kappa_j(\hat\xi)} \quad \text{for any } \xi \in \partial\Omega, \tag{2.6}$$

where $\xi^* = \xi + \frac{R}{2}\nu(\xi) \in \Gamma^*$ and $\hat\xi = \xi + R\nu(\xi) \in \Gamma$. Let $\mu = cR^{N-1}$. Then, it follows from (2.3) and (2.6) that

$$\prod_{j=1}^{N-1}(1-R\kappa_j) = \mu, \ \prod_{j=1}^{N-1}(1+R\hat\kappa_j) = \frac{1}{\mu}, \ \text{and} \ \prod_{j=1}^{N-1}\frac{1-\frac{R}{2}\kappa_j^*}{1+\frac{R}{2}\kappa_j^*} = \mu. \tag{2.7}$$

We distinguish three cases:

$$\text{(I) } \mu > 1, \ \text{(II) } \mu < 1, \ \text{and (III) } \mu = 1.$$

Let us consider case (I) first. By the arithmetic-geometric mean inequality and the first equation in (2.7) we have

$$1 - RH_{\partial\Omega} = \frac{1}{N-1}\sum_{j=1}^{N-1}(1-R\kappa_j) \ge \left\{\prod_{j=1}^{N-1}(1-R\kappa_j)\right\}^{\frac{1}{N-1}} = \mu^{\frac{1}{N-1}} > 1.$$

This shows that

$$H_{\partial\Omega} \le -\frac{1}{R}\left(\mu^{\frac{1}{N-1}} - 1\right) < 0. \tag{2.8}$$

Since

$$(N-1)H_{\partial\Omega} = \ \mathrm{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right) \ \text{ in } \mathbb{R}^{N-1},$$

by using the divergence theorem we get a contradiction as in the proof of [MS2, Theorem 3.3]. In case (II), by the arithmetic-geometric mean inequality and the second equation in (2.7) we have

$$1 + RH_\Gamma = \frac{1}{N-1}\sum_{j=1}^{N-1}(1+R\hat\kappa_j) \ge \left\{\prod_{j=1}^{N-1}(1+R\hat\kappa_j)\right\}^{\frac{1}{N-1}} = \mu^{-\frac{1}{N-1}} > 1.$$

This shows that

$$H_\Gamma \geq \frac{1}{R}\left(\mu^{-\frac{1}{N-1}} - 1\right) > 0, \tag{2.9}$$

which yields a contradiction similarly.

Thus, it remains to consider case (III). By (2.8) and (2.9), we have

$$H_{\partial\Omega} \leq 0 \leq H_\Gamma. \tag{2.10}$$

Let us consider case (i) of Theorem 1.1 first. Since $N = 3$ and $\mu = 1$, it follows from the third equation of (2.7) that

$$2H_{\Gamma^*} = \kappa_1^* + \kappa_2^* = 0.$$

We observe that Γ^* is the entire graph of a function on $\mathbb{R}^2$. Therefore, by the Bernstein's theorem for the minimal surface equation, Γ^* must be a hyperplane. This gives the conclusion desired. (See [GT, Giu] for Bernstein's theorem.)

Secondly, we consider case (iii) of Theorem 1.1. Take any point $\xi \in A$. If all the κ_j's are non-positive at ξ, then they must vanish at ξ, since $\prod_{j=1}^{N-1}(1 - R\kappa_j) = 1$. On the other hand, we have that

$$1 - RH_{\partial\Omega} = \frac{1}{N-1}\sum_{j=1}^{N-1}(1 - R\kappa_j) \geq \left\{\prod_{j=1}^{N-1}(1 - R\kappa_j)\right\}^{\frac{1}{N-1}} = 1;$$

thus, if $H_{\partial\Omega} \geq 0$ at ξ, then all the κ_j must be equal to each other and hence again they must vanish at ξ. Since $\xi \in A$ is arbitrary, we have

$$\kappa_j \equiv 0 \ \text{ on } A \ \text{ for every } j = 1, \cdots, N-1,$$

and hence φ is affine on A. Then by the analyticity of φ we see that φ is affine on the whole of $\mathbb{R}^{N-1}$. This shows that $\partial\Omega$ must be a hyperplane.

Thus it remains to consider case (ii) of Theorem 1.1. In this case, there exists a constant $L \geq 0$ satisfying

$$\sup_{\mathbb{R}^{N-1}} |\nabla\varphi| = L < \infty.$$

Then, it follows from (1) and (3) of Lemma 2.1 that

$$\sup_{\mathbb{R}^{N-1}} |\nabla \psi| = \sup_{\mathbb{R}^{N-1}} |\nabla \varphi| = L < \infty. \tag{2.11}$$

Hence, in view of this and (3) of Lemma 2.1, we can define a number $K^* > 0$ by

$$K^* = \inf\{K > 0 : \psi \le \varphi + K \ \text{ in } \mathbb{R}^{N-1}\}. \tag{2.12}$$

Then we have

$$\varphi \le \psi \le h \ \text{ in } \mathbb{R}^{N-1}, \tag{2.13}$$

where $h : \mathbb{R}^{N-1} \to \mathbb{R}$ is defined by

$$h(x') = \varphi(x') + K^* \ \text{ for } x' \in \mathbb{R}^{N-1}.$$

Moreover, by writing

$$M(h) = \operatorname{div}\left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right) \text{ and } M(\psi) = \operatorname{div}\left(\frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}}\right),$$

from (2.10) and (2.13) we have

$$M(h) \le 0 \le M(\psi) \ \text{ and } \ \psi \le h \ \text{ in } \mathbb{R}^{N-1}. \tag{2.14}$$

Hence, the method of sub- and super-solutions with the help of (2.11) yields that there exists $v \in C^\infty(\mathbb{R}^{N-1})$ satisfying

$$M(v) = 0 \ \text{ and } \ \psi \le v \le h \ \text{ in } \mathbb{R}^{N-1}, \text{ and } \sup_{\mathbb{R}^{N-1}} |\nabla v| < \infty. \tag{2.15}$$

Indeed, take a sequence of balls $\{B_n(0)\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{N-1}$ and consider the boundary value problem for each $n \in \mathbb{N}$:

$$M(v) = 0 \ \text{ in } \ B_n(0) \ \text{ and } \ v = \psi \ \text{ on } \partial B_n(0). \tag{2.16}$$

By [GT, Theorem 16.9], for each $n \in \mathbb{N}$ there exists a C^2-function v_n on $\overline{B_n(0)}$ solving problem (2.16). In view of (2.14), it then follows from the comparison principle that

$$\psi \le v_n \le h \quad \text{in } \ B_n(0) \quad \text{for every } n \in \mathbb{N}. \tag{2.17}$$

Therefore, with the help of the interior estimates for the minimal surface equation (see [GT, Corollary 16.7]), we prove that v_n belongs to $C^\infty(B_n(0))$ and for every $\rho > 0$ and every $k \in \mathbb{N}$, the C^k norms of $\{v_n\}_{n>\rho}$ are bounded. In conclusion, the Cantor diagonal process together with Arzela-Ascoli theorem yields a solution $v \in C^\infty(\mathbb{R}^{N-1})$ of (2.15). It remains to show that ∇v is bounded in $\mathbb{R}^{N-1}$. For this purpose, we define a sequence of C^∞ functions $\{w_n\}$ on $\overline{B_1(0)}$ by

$$w_n(x') = \frac{1}{n}\, v_n(nx') \quad \text{for} \quad x' \in B_1(0) \quad \text{and for every } n \in \mathbb{N}. \tag{2.18}$$

Then, each w_n satisfies

$$M(w_n) = 0 \quad \text{in} \quad B_1(0) \quad \text{and} \quad w_n(x') = \frac{1}{n}\psi(nx') \quad \text{for } x' \in \partial B_1(0). \tag{2.19}$$

Since $|(\nabla\psi)(nx')| \le L$, we have $|\psi(nx')| \le |\psi(0)| + n|x'|L$. Therefore, it follows from the maximum principle that

$$\max_{B_1(0)} |w_n| \le \max_{\partial B_1(0)} \frac{1}{n}|\psi(nx')| \le |\psi(0)| + L \quad \text{for every } n \in \mathbb{N}.$$

Hence, by [GT, Corollary 16.7], in particular there exists a constant C satisfying

$$|\nabla w_n(x')| \le C \quad \text{for every } x' \in B_{\frac{1}{2}}(0) \quad \text{and for every } n \in \mathbb{N}.$$

By observing that $\nabla w_n(x') = (\nabla v_n)(nx')$, we see that

$$|\nabla v_n| \le C \quad \text{in } B_{n/2}(0) \quad \text{for every } n \in \mathbb{N},$$

and hence

$$|\nabla v| \le C \quad \text{in} \quad \mathbb{R}^{N-1}, \tag{2.20}$$

which shows that the last claim in (2.15) holds. Therefore, Moser's theorem [Mo, Corollary, p. 591] implies that v is affine. We set $\eta = \nabla v \in \mathbb{R}^{N-1}$.

On the other hand, by the definition of K^* in (2.12), there exists a sequence $\{z_n\}$ in $\mathbb{R}^{N-1}$ satisfying

$$\lim_{n\to\infty} (h(z_n) - \psi(z_n)) = 0. \tag{2.21}$$

Define a sequence of functions $\{\varphi_n\}$ by

$$\varphi_n(x') = h(x'+z_n) - h(z_n) \ \ (= \varphi(x'+z_n) - \varphi(z_n))\,.$$

Note that the principal curvatures $\kappa_1, \cdots, \kappa_{N-1}$ of $\partial\Omega$ are the eigenvalues of the real symmetric matrix $G^{-\frac{1}{2}}BG^{-\frac{1}{2}}$, where the matrices G and B have entries

$$G_{ij} = \delta_{ij} + \frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j} \ \text{ and } \ B_{ij} = \frac{1}{\sqrt{1+|\nabla\varphi|^2}}\frac{\partial^2\varphi}{\partial x_i \partial x_j}, \tag{2.22}$$

for $i, j = 1, \ldots, N-1$, and δ_{ij} is Kronecker's symbol (see [R, Proposition 3.1]).

Then from (2.2) and (2.11) we see that all the second derivatives of φ are bounded in $\mathbb{R}^{N-1}$. Hence we can conclude that there exists a subsequence $\{\varphi_{n'}\}$ of $\{\varphi_n\}$ and a function $\varphi_\infty \in C^1(\mathbb{R}^{N-1})$ such that $\varphi_{n'} \to \varphi_\infty$ in $C^1(\mathbb{R}^{N-1})$ as $n' \to \infty$. Since $M(\varphi_n) \le 0$ in $\mathbb{R}^{N-1}$, we have that $M(\varphi_\infty) \le 0$ in $\mathbb{R}^{N-1}$ in the weak sense. Also, since $0 \le h(x'+z_{n'}) - v(x'+z_{n'})$ in $\mathbb{R}^{N-1}$, with the help of (2.21), letting $n' \to \infty$ yields that

$$0 \le \varphi_\infty(x') - \eta\cdot x' \ \text{ in } \mathbb{R}^{N-1}.$$

Consequently, we have

$$\begin{aligned} &M(\varphi_\infty) \le 0 = M(\eta\cdot x') \ \text{ and } \varphi_\infty(x') \ge \eta\cdot x' \ \text{ in } \mathbb{R}^{N-1}, \\ &\text{and } \varphi_\infty(0) = 0 = \eta\cdot 0. \end{aligned}$$

Hence, the strong comparison principle implies that $\varphi_\infty(x') \equiv \eta\cdot x'$ in $\mathbb{R}^{N-1}$. Here we have used Theorem 10.7 together with Theorem 8.19 in [GT]. Therefore we conclude that as $n \to \infty$

$$\varphi(x'+z_n) - (v(x'+z_n) - K^*) \to 0 \ \text{ in } C^1(\mathbb{R}^{N-1}). \tag{2.23}$$

Similarly, we can obtain that as $n \to \infty$

$$v(x'+z_n) - \psi(x'+z_n) \to 0 \ \text{ in } C^1(\mathbb{R}^{N-1}). \tag{2.24}$$

Indeed, it follows from (2.2) and (2.3) that there exists a positive constant $\tau > 0$ such that for each $j = 1, \cdots, N-1$

$$-\frac{1}{r_0} \le \kappa_j(\xi) \le \frac{1}{R} - \tau \quad \text{for every } \xi \in \partial\Omega.$$

Combining this with (2.6) yields that all the principal curvatures $\hat{\kappa}_1, \cdots, \hat{\kappa}_{N-1}$ of Γ are bounded. Then, in view of this fact and the relationship between the function ψ and the principal curvatures $\hat{\kappa}_1, \cdots, \hat{\kappa}_{N-1}$, from (2.11) we see that all the second derivatives of ψ are bounded in $\mathbb{R}^{N-1}$. Thus we can obtain (2.24) by the same argument as in proving (2.23).

Therefore, it follows from (3) of Lemma 2.1, (2.23), and (2.24) that the distance between two hyperplanes determined by two affine functions v and $v - K^*$ must be R. Hence, since $v - K^* \le \varphi \le \psi \le v$ in $\mathbb{R}^{N-1}$, we conclude that

$$\psi \equiv v \quad \text{and} \quad \varphi \equiv v - K^* \quad \text{in } \mathbb{R}^{N-1},$$

which shows that $\partial\Omega$ is a hyperplane. $\Box$

3 Concluding remarks

Let us explain the relationship between Theorem 1.1 and Theorems 3.2, 3.3, and 3.4 in [MS2]. When $\mu = 1$, we have

$$1 + RH_\Gamma = \frac{1}{N-1}\sum_{j=1}^{N-1}(1 + R\hat{\kappa}_j) \ge \left\{\prod_{j=1}^{N-1}(1 + R\hat{\kappa}_j)\right\}^{\frac{1}{N-1}} = 1.$$

Therefore, the assumption, $H_\Gamma \le 0$, of [MS2, Theorem 3.2] implies that $\hat{\kappa}_j \equiv 0$ for every $j = 1, \cdots, N-1$. This shows that Γ is a hyperplane, and hence $\partial\Omega$ must be a hyperplane. Thus, [MS2, Theorem 3.2] is contained in Theorem 1.1 with its proof. In the case where Ω is given by (1.4), [MS2, Theorem 3.3] is contained in Theorem 1.1 with condition (iii). Since [MS2, Theorem 3.4] does not assume the uniform exterior sphere condition for Ω, it is independent of Theorem 1.1.

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